Brownian motion and Stochastic Calculus
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Assignment 11—solutions

Exercise 1

Let B be an $(\mathbb{F},\mathbb{P})$–Brownian motion and M an $(\mathbb{F},\mathbb{P})$-martingale such that $\mathrm{d}M_t = \sigma M_t \mathrm{d}B_t$ with $\sigma > 0$ given and $M_0 = 1$.

1) Give the Itô decomposition of $Y_t := (M_t)^{-1}$, $t \geq 0$.

2) Let $\mathbb{Q}$ be the probability measure defined by $\mathrm{d}\mathbb{Q}/\mathrm{d}\mathbb{P} := M$. What can you say about the law of Y under $\mathbb{Q}$?

3) Let $K \geq 0$ be given. Show that

$$\mathbb{E}^{\mathbb{P}}\big[(M_T - K)^+\big] = K\mathbb{E}^{\mathbb{P}}\bigg[\bigg(\frac{1}{K} - M_T\bigg)^+\bigg].$$

1) **It suffices to apply Itô's formula, noticing also that** $M_t = \mathcal{E}(\sigma B)_t$

$$\mathrm{d}Y_t = -\frac{1}{M_t^2}\mathrm{d}M_t + \frac{1}{M_t^3}\sigma^2 M_t^2 \mathrm{d}t = -\sigma Y_t \mathrm{d}B_t + \sigma^2 Y_t \mathrm{d}t.$$

2) **First, Novikov's condition gives us immediately that** M **is a martingale, and we can use it as a change of measure, say at least on** $\mathcal{F}_T$**. Then by Girsanov's theorem (and the symmetry of Brownian motion)**

$$B_t^{\mathbb{Q}} := -B_t + \sigma t,$$

is an $(\mathbb{F},\mathbb{Q})$–**Brownian motion, so that**

$$\mathrm{d}Y_t = \sigma Y_t \mathrm{d}B_t^{\mathbb{Q}},$$

and thus the law of Y **under** $\mathbb{Q}$ **is the same as the law of** M **under** $\mathbb{P}$**.**

3) **We have**

$$\mathbb{E}^{\mathbb{P}}\big[(M_T - K)^+\big] = K\mathbb{E}^{\mathbb{P}}\big[M_T(K^{-1} - Y_T)^+\big] = K\mathbb{E}^{\mathbb{Q}}\big[(K^{-1} - Y_T)^+\big] = K\mathbb{E}^{\mathbb{P}}\big[(K^{-1} - M_T)^+\big].$$

Exercise 2

The goal of this question is to prove Novikov's condition which gives a sufficient requirement for an exponential (local) martingale to be a uniformly integrable martingale. Let $\mathbb{F} := (\mathcal{F}_t)_{t\geq 0}$ be a filtration satisfying the usual conditions. When N is a continuous $(\mathbb{F},\mathbb{P})$–local martingale we will write $\mathcal{E}(N) := \exp(N - [N]/2)$. Suppose that M is a given continuous $(\mathbb{F},\mathbb{P})$–local martingale with $M_0 = 0$, $\mathbb{P}$–a.s.

1) Prove that $\mathcal{E}(M)$ is an $(\mathbb{F},\mathbb{P})$–super-martingale. Moreover, show that if $\mathbb{E}^{\mathbb{P}}[\mathcal{E}(M)_\infty] = 1$, then $\mathcal{E}(M)$ is a $\mathbb{P}$–uniformly integrable $(\mathbb{F},\mathbb{P})$-martingale.

2) Consider $p \geq 1$, $\varepsilon \in (0,1)$, $\eta \in (0,1)$ and $\rho \in \mathbb{R}$. Prove that when M is bounded by a deterministic constant, then for any $t \geq 0$

$$\mathbb{E}^{\mathbb{P}}\bigg[\sup_{s\in[0,\infty)} \mathcal{E}(\eta M)_s^p\bigg] \leq \bigg(\frac{p}{p-1}\bigg)^p \sup_{s\in[0,\infty)} \mathbb{E}^{\mathbb{P}}\big[\mathcal{E}(\eta M)_s^p\big], \text{ for } p > 1,$$
$$\mathbb{E}^{\mathbb{P}}\Big[\big(\mathrm{e}^{\eta M_t - [\eta M]_t/2}\big)^p\Big] \leq \mathbb{E}^{\mathbb{P}}\Big[\big(\mathrm{e}^{(\rho-p)[\eta M]_t/2}\big)^{1/\varepsilon}\Big]^{\varepsilon}, \text{ for } \rho = p^2/(1-\varepsilon).$$

3) Use 2) and a localisation argument to establish the following: If $\mathbb{E}^{\mathbb{P}}[\mathrm{e}^{[M]_\infty/2}] < +\infty$ (called Novikov's condition) then for all $\eta \in (0,1)$, there exists $p > 1$ such that

$$\mathbb{E}^{\mathbb{P}}\left[\sup_{t\in[0,\infty)} \mathcal{E}(\eta M)^p\right] < \infty, \text{ and hence } \mathbb{E}^{\mathbb{P}}\left[\sup_{t\in[0,\infty)} \mathcal{E}(\eta M)\right] < +\infty.$$

Deduce that then $\mathcal{E}(\eta M)$ is a $\mathbb{P}$–uniformly integrable $(\mathbb{F},\mathbb{P})$-martingale, so that $\mathbb{E}^{\mathbb{P}}[\mathcal{E}(\eta M)_t] = 1$, for all $t \in [0,\infty]$.

4) Using 3) and part of the argument given in 2), show that (again assuming Novikov's condition) for $\varepsilon \in (0,1)$

$$1 = \mathbb{E}^{\mathbb{P}}[\mathcal{E}(\eta M)_\infty] \leq \mathbb{E}^{\mathbb{P}}[\mathcal{E}(M)_\infty]^{1-\varepsilon}\mathbb{E}^{\mathbb{P}}\left[\mathrm{e}^{(1-\varepsilon)[M]_\infty/2}\right]^{\varepsilon} \leq \mathbb{E}^{\mathbb{P}}[\mathcal{E}(M)_\infty]^{1-\varepsilon}\mathbb{E}^{\mathbb{P}}\left[\mathrm{e}^{[M]_\infty/2}\right]^{\varepsilon}, \text{ where } \eta := 1-\varepsilon.$$

5) Combine the above results to deduce that under Novikov's condition, $\mathcal{E}(M)$ is a $\mathbb{P}$–uniformly integrable $(\mathbb{F},\mathbb{P})$-martingale.

1) **First of all, we know that $\mathcal{E}(M)$ is a local martingale. Let us define stopping times $\tau_n = \inf\{t \geq 0 : |M_t| \geq n\}$ for $n \in \mathbb{N}$. Then $\mathcal{E}(M)^{\tau_n}$ are local martingales that are each bounded by a deterministic constant, so they are all martingales. By Fatou's lemma, we get that for $t \geq 0$,**

$$0 \leq \mathbb{E}^{\mathbb{P}}[\mathcal{E}(M)_t] = \mathbb{E}^{\mathbb{P}}\left[\liminf_{n\to\infty} \mathcal{E}(M)_t^{\tau_n}\right] \leq \liminf_{n\to\infty} \mathbb{E}^{\mathbb{P}}[\mathcal{E}(M)_t^{\tau_n}] = 1.$$

This yields integrability. Clearly $\mathcal{E}(M)$ is adapted. Consider now $0 \leq s \leq t$. Then

$$\begin{aligned}\mathcal{E}(M)_s &= \liminf_{n\to\infty} \mathcal{E}(M)_s^{\tau_n} = \liminf_{n\to\infty} \mathbb{E}^{\mathbb{P}}\left[\mathcal{E}(M)_t^{\tau_n}\middle|\mathcal{F}_s\right] \\ &\geq \mathbb{E}^{\mathbb{P}}\left[\liminf_{n\to\infty} \mathcal{E}(M)_t^{\tau_n}\middle|\mathcal{F}_s\right] = \mathbb{E}^{\mathbb{P}}[\mathcal{E}(M)_t|\mathcal{F}_s], \ \mathbb{P}\text{–a.s.}\end{aligned}$$

So $\mathcal{E}(M)$ is a (non-negative) supermartingale as required. We saw above that $\mathcal{E}(M)$ is bounded in $\mathbb{L}^1(\mathbb{R},\mathcal{F},\mathbb{P})$, so $\mathcal{E}(M)_t$ converges, $\mathbb{P}$–a.s., to an (integrable) limit $\mathcal{E}(M)_\infty$ as $t \to \infty$. Assume now that $\mathbb{E}^{\mathbb{P}}[\mathcal{E}(M)_\infty] = 1$. It will suffice to prove that for $t \geq 0$

$$\mathcal{E}(M)_t = \mathbb{E}^{\mathbb{P}}[\mathcal{E}(M)_\infty|\mathcal{F}_t], \ \mathbb{P}\text{–a.s.}$$

By Fatou's lemma, we see that $\mathcal{E}(M)_t \geq \mathbb{E}^{\mathbb{P}}[\mathcal{E}(M)_\infty|\mathcal{F}_t]$, $\mathbb{P}$–a.s. and hence

$$\mathbb{E}^{\mathbb{P}}\left[\left|\mathcal{E}(M)_t - \mathbb{E}^{\mathbb{P}}[\mathcal{E}(M)_\infty|\mathcal{F}_t]\right|\right] = \mathbb{E}^{\mathbb{P}}\left[\mathcal{E}(M)_t - \mathbb{E}^{\mathbb{P}}[\mathcal{E}(M)_\infty|\mathcal{F}_t]\right] = \mathbb{E}^{\mathbb{P}}[\mathcal{E}(M)_t] - \mathbb{E}^{\mathbb{P}}[\mathcal{E}(M)_\infty] \leq 1 - \mathbb{E}^{\mathbb{P}}[\mathcal{E}(M)_\infty] = 0,$$

which immediately implies the claim.

2) **Since M is bounded by a deterministic constant, so is $\mathcal{E}(\eta M)$ and hence $\mathcal{E}(\eta M)$ is a martingale. The first claim is thus an immediate application of Doob's inequality. For the second inequality, let us write**

$$\left(\mathrm{e}^{\eta M_t - [\eta M]_t/2}\right)^p = \mathrm{e}^{p\eta M_t - \rho[\eta M]_t/2}\mathrm{e}^{(\rho-p)[\eta M]_t/2}.$$

Applying Hölder's inequality with exponents $1/(1-\varepsilon)$ and $1/\varepsilon$ yields

$$\mathbb{E}^{\mathbb{P}}\left[\left(\mathrm{e}^{\eta M_t - [\eta M]_t/2}\right)^p\right] \leq \mathbb{E}^{\mathbb{P}}\left[\left(\mathrm{e}^{p\eta M_t - \rho[\eta M]_t/2}\right)^{1/(1-\epsilon)}\right]^{1-\varepsilon}\mathbb{E}^{\mathbb{P}}\left[\left(\mathrm{e}^{(\rho-p)[\eta M]_t/2}\right)^{1/\varepsilon}\right]^{\varepsilon}.$$

If $\rho = p^2/(1-\varepsilon)$, we see that

$$\mathbb{E}^{\mathbb{P}}\left[\left(\mathrm{e}^{p\eta M_t - \rho[\eta M]_t/2}\right)^{1/(1-\varepsilon)}\right] = \mathbb{E}^{\mathbb{P}}[\mathcal{E}(\eta' M)_t] = 1, \textbf{ with } \eta' := \eta p/(1-\varepsilon),$$

since $\mathcal{E}(\eta' M)$ is a martingale starting from 1. Combining the results in the two previous displays thence yields the claim.

3) **Fix $\eta \in (0,1)$. In 2), we showed that for $p>1$ and $\varepsilon \in (0,1)$,**

$$\mathbb{E}^{\mathbb{P}}\bigg[\sup_{t\in[0,\infty)} \mathcal{E}(\eta M)_t^p\bigg] \leq \bigg(\frac{p}{p-1}\bigg)^p \mathbb{E}^{\mathbb{P}}\bigg[\exp\bigg(\frac{p^2/(1-\varepsilon)-p}{\epsilon}\eta^2\frac{[M]_\infty}{2}\bigg)\bigg],$$

whenever M is bounded by a deterministic constant. To see that this also holds without the boundedness assumption, we may apply the above result to M^{τ_n} to see that

$$\mathbb{E}^{\mathbb{P}}\bigg[\sup_{t\in[0,\infty)} \big(\mathcal{E}(\eta M)_t^{\tau_n}\big)^p\bigg] \leq \bigg(\frac{p}{p-1}\bigg)^p \mathbb{E}^{\mathbb{P}}\bigg[\exp\bigg(\frac{p^2/(1-\varepsilon)-p}{\varepsilon}\eta^2\frac{[M]_{\tau_n}}{2}\bigg)\bigg]$$
$$\leq \bigg(\frac{p}{p-1}\bigg)^p \mathbb{E}^{\mathbb{P}}\bigg[\exp\bigg(\frac{p^2/(1-\varepsilon)-p}{\varepsilon}\eta^2\frac{[M]_\infty}{2}\bigg)\bigg],$$

and letting $n\to\infty$ (using monotone convergence) yields the required generalisation.

We will now show that we can take $p>1$ and $\varepsilon\in(0,1)$ such that

$$\frac{p^2/(1-\varepsilon)-p}{\varepsilon}\eta^2 \leq 1.$$

One way to see this is to write $p=1+\delta$ and Taylor expand to obtain

$$\frac{p^2/(1-\varepsilon)-p}{\varepsilon}\eta^2 = \frac{(1+2\delta+O(\delta^2))(1+\varepsilon+O(\varepsilon^2))-1-\delta}{\varepsilon}\eta^2 = (1+2\delta+O(\varepsilon)+O(\delta^2)/\varepsilon)\eta^2, \text{ as } \varepsilon \text{ and } \delta \text{ go to } 0.$$

Since $\eta^2<1$, the above inequality can therefore be satisfied by first choosing $\varepsilon>0$ and then $\delta>0$ sufficiently small. For these values of $p=1+\delta$ and $\varepsilon\in(0,1)$, therefore

$$\mathbb{E}^{\mathbb{P}}\bigg[\sup_{t\in[0,\infty)} \mathcal{E}(\eta M)_t^p\bigg] \leq \big(p/(p-1)\big)^p \mathbb{E}^{\mathbb{P}}\big[\mathrm{e}^{[M]_\infty/2}\big] < \infty.$$

In particular, $\mathcal{E}(\eta M)$ is a uniformly integrable martingale and $\mathbb{E}^{\mathbb{P}}[\mathcal{E}(\eta M)_t] = \mathbb{E}^{\mathbb{P}}[\mathcal{E}(\eta M)_0]=1$ for all $t\in[0,\infty]$.

4) **Firstly note that since $\mathbb{E}^{\mathbb{P}}[\mathrm{e}^{[M]_\infty/2}]<\infty$, $\mathbb{E}^{\mathbb{P}}[[M]_\infty]<\infty$ and hence M is an $\mathbb{L}^2(\mathbb{R},\mathcal{F},\mathbb{P})$-bounded martingale. Consider $\eta=1-\varepsilon$ and $\rho=1/(1-\varepsilon)$ for $\varepsilon\in(0,1)$. Then using the same argument as in 2), we see that**

$$\mathbb{E}^{\mathbb{P}}[\mathcal{E}(\eta M)_\infty] = \mathbb{E}^{\mathbb{P}}\big[\mathrm{e}^{\eta M_\infty - [\eta M]_\infty/2}\big] \leq \mathbb{E}^{\mathbb{P}}\Big[\big(\mathrm{e}^{\eta M_\infty - \rho[\eta M]_\infty/2}\big)^{1/(1-\varepsilon)}\Big]^{1-\varepsilon}\mathbb{E}^{\mathbb{P}}\Big[\big(\mathrm{e}^{(\rho-1)[\eta M]_\infty/2}\big)^{1/\varepsilon}\Big]^{\varepsilon}$$
$$= \mathbb{E}^{\mathbb{P}}[\mathcal{E}(M)_\infty]^{1-\varepsilon}\mathbb{E}^{\mathbb{P}}\big[\mathrm{e}^{(1-\varepsilon)[M]_\infty/2}\big]^{\varepsilon} \leq \mathbb{E}^{\mathbb{P}}[\mathcal{E}(M)_\infty]^{1-\varepsilon}\mathbb{E}^{\mathbb{P}}\big[\mathrm{e}^{[M]_\infty/2}\big]^{\varepsilon}.$$

We conclude by noting that $\mathbb{E}^{\mathbb{P}}[\mathcal{E}(\eta M)_\infty]=1$ as derived in part 3).

5) **Finally, by taking $\epsilon\to 0$ in the inequality**

$$1 \leq \mathbb{E}(E(M)_\infty)^{1-\epsilon}\,\mathbb{E}\left(e^{\langle M\rangle_\infty/2}\right)^\epsilon$$

that was derived in (iv), we get $\mathbb{E}^{\mathbb{P}}[\mathcal{E}(M)_\infty]\geq 1$. By the argument in (i) also $\mathbb{E}^{\mathbb{P}}[\mathcal{E}(M)_\infty]\leq 1$ and hence $\mathbb{E}^{\mathbb{P}}[\mathcal{E}(M)_\infty]=1$. The result then follows from the second part of 1).

Exercise 3

Let B be a standard Brownian motion (in some filtration satisfying the usual conditions). Fix $t\geq 0$. The goal of this exercise is to compute the moment generating function of $\int_0^t B_s^2\mathrm{d}s$. To this end, fix $\kappa>0$ and $t\geq 0$.

1) Show that the process D defined by

$$D_s^t := \exp\bigg(-\kappa\int_0^{s\wedge t} B_u\mathrm{d}B_u - \frac{\kappa^2}{2}\int_0^{s\wedge t} B_u^2\mathrm{d}u\bigg),\ s\geq 0,$$

is a $\mathbb{P}$–uniformly integrable $(\mathbb{F},\mathbb{P})$-martingale. Moreover, observe that $\int_0^s B_u\mathrm{d}B_u = (B_s^2-s)/2$, $\mathbb{P}$–a.s., for all $s\geq 0$.

2) Now we define a new probability measure $\mathbb{Q}$ via $\mathrm{d}\mathbb{Q}/\mathrm{d}\mathbb{P} := D^t_\infty$. Prove that under the measure $\mathbb{Q}$, the process

$$W^t_s := B_s + \kappa \int_0^{s\wedge t} B_u \mathrm{d}u,$$

is a standard Brownian motion (in the given filtration). Deduce that under $\mathbb{Q}$, $B_t \sim N\big(0, (1-\mathrm{e}^{-2\kappa t})/(2\kappa)\big)$. Use this to prove that

$$\mathbb{E}^{\mathbb{P}}\Big[\mathrm{e}^{-\frac{\kappa^2}{2}\int_0^t B_u^2 \mathrm{d}u}\Big] = \mathbb{E}^{\mathbb{Q}}\Big[\mathrm{e}^{\frac{\kappa}{2}(B_t^2 - t)}\Big] = \frac{1}{\sqrt{\cosh(\kappa t)}}.$$

3) Let $\widetilde{B}$ be another standard Brownian motion, independent of B. Show that

$$\int_0^t \big(B_u^2 + \widetilde{B}_u^2\big)\mathrm{d}u \overset{\text{law}}{=} \inf\{s \geq 0 \colon |B_s| = t\}.$$

Is it true that $\int_0^\cdot \big(B_u^2 + \widetilde{B}_u^2\big)\mathrm{d}u \overset{\text{law}}{=} \inf\{s \geq 0 \colon |B_s| = \cdot\}$?

1) **Let us consider the local martingale $M = -\kappa\int_0^{\cdot\wedge t} B_s \mathrm{d}B_s$ so that $D = \exp(M - [M]/2)$, $\mathbb{P}$–a.s. We first observe that second statement $\int_0^s B_r \mathrm{d}B_r = (B_s^2 - s)/2$ for all $s \geq 0$, $\mathbb{P}$–a.s., is a consequence of Itô's formula. Moreover**

$$0 \leq D_s \leq \exp\bigg(-\kappa\int_0^{s\wedge t} B_r \mathrm{d}B_r\bigg) = \exp\Big(\kappa/2(s\wedge t - B^2_{s\wedge t})\Big) \leq \mathrm{e}^{\kappa t/2},$$

for all $s \geq 0$. Therefore D is bounded by a deterministic constant and hence a uniformly integrable martingale.

2) **By Girsanov's theorem $W = B - [B, M]$ is a local martingale under $\mathbb{Q}$. Moreover, we have $[W]_s = s$ for all $s \geq 0$, $\mathbb{P}$–a.s. (under $\mathbb{P}$ and $\mathbb{Q}$), so by Lévy's characterisation, W is a standard Brownian motion in the given filtration. Also**

$$[B, M] = -\kappa\int_0^{\cdot\wedge t} B_s \mathrm{d}s,\ \mathbb{P}\text{–}\textbf{a.s. (w.r.t. both } \mathbb{P} \textbf{ and } \mathbb{Q}\textbf{)}.$$

We are now working with respect to $\mathbb{Q}$. Let B' be the unique strong solution to the following SDE (again noting that W is a standard Brownian motion in the given filtration)

$$\mathrm{d}B'_s = \mathrm{d}W_s - \kappa B'_s \mathrm{d}s,\ B'_0 = 0,\ \mathbb{P}\text{–}\textbf{a.s. for } t \geq 0.$$

This is an Ornstein–Uhlenbeck process, so we know that $B'_t \sim N\big(0, (1-\mathrm{e}^{-2\kappa t})/(2\kappa)\big)$. It thus suffices to prove that $B'_t = B_t$, $\mathbb{P}$–a.s.. To see this, observe that

$$B'_s - B_s = -\kappa\int_0^s (B'_r - B_r)\mathrm{d}r,\ \textbf{for all } s \leq t\textbf{, } \mathbb{P}\text{–}\textbf{a.s.}.$$

By an application of Gronwall's lemma, we can deduce that $B = B'$ as required. Using this, we can compute

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}}\Big[\mathrm{e}^{\frac{\kappa}{2}(B_t^2 - t)}\Big] &= \int_{\mathbb{R}} \frac{\mathrm{e}^{-x^2/2}}{\sqrt{2\pi}} \exp\bigg(\frac{\kappa}{2}\Big(\frac{1-\mathrm{e}^{-2\kappa t}}{2\kappa}x^2 - t\Big)\bigg)\mathrm{d}x \\ &= \mathrm{e}^{-\kappa t/2}\int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} \exp\bigg(-\frac{x^2}{2}\frac{1+\mathrm{e}^{-2\kappa t}}{2}\bigg) \\ &= \frac{\mathrm{e}^{-\kappa t/2}}{\sqrt{(1+\mathrm{e}^{-2\kappa t})/2}} = \frac{1}{\sqrt{\cosh(\kappa t)}},\end{aligned}$$

as required. It only remains to express the expectation w.r.t. $\mathbb{Q}$ on the left-hand side in terms of an expectation w.r.t. $\mathbb{P}$. Using the definition of $\mathbb{Q}$ we get that

$$\mathbb{E}^{\mathbb{Q}}\Big[\mathrm{e}^{\frac{\kappa}{2}(B_t^2-t)}\Big] = \mathbb{E}^{\mathbb{P}}\bigg[\exp\bigg(\frac{\kappa}{2}(B_t^2-t) - \kappa\int_0^t B_s\mathrm{d}B_s - \frac{\kappa^2}{2}\int_0^t B_s^2\mathrm{d}s\bigg)\bigg]$$
$$= \mathbb{E}^{\mathbb{P}}\Big[\mathrm{e}^{-\frac{\kappa^2}{2}\int_0^t B_s^2\mathrm{d}s}\Big].$$

3) **The moment generating function of** $\inf\{s \geq 0\colon |B_s| = t\}$ **is known, and for** $\kappa > 0$

$$\mathbb{E}^{\mathbb{P}}\Big[\mathrm{e}^{-\frac{\kappa^2}{2}\inf\{s\geq 0\colon |B_s|=t\}}\Big] = \frac{1}{\cosh(\kappa t)} = \mathbb{E}^{\mathbb{P}}\Big[\mathrm{e}^{-\frac{\kappa^2}{2}\int_0^t B_s^2\mathrm{d}s}\Big]\mathbb{E}^{\mathbb{P}}\Big[\mathrm{e}^{-\frac{\kappa^2}{2}\int_0^t \tilde{B}_s^2\mathrm{d}s}\Big]$$
$$= \mathbb{E}^{\mathbb{P}}\Big[\mathrm{e}^{-\frac{\kappa^2}{2}\int_0^t (B_s^2+\tilde{B}_s^2)\mathrm{d}s}\Big].$$

If the moment generating functions of two non-negative random variables agree, they have the same law, so the assertion follows. Finally, it is not true that

$$\int_0^{\cdot} (B_s^2 + \tilde{B}_s^2)\mathrm{d}s \overset{d}{=} \inf\{s \geq 0\colon |B_s| = \cdot\},$$

since the process on the left-hand side is continuous while the process on the right-hand side has jump discontinuities a.s. (we may observe that it is increasing and is left-continuous however).

Exercise 4

Fix a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$. Let B be an $(\mathbb{F}, \mathbb{P})$–Brownian motion, μ a bounded $\mathbb{F}$-adapted and measurable process, and fix some $x_0 \in \mathbb{R}$.

1) Show that there exists a unique solution to the SDE

$$X_t = x_0 + \int_0^t \mu_s\mathrm{d}s + \int_0^t X_s\mathrm{d}B_s,\ t \geq 0,$$

which is given by

$$X_t = \mathcal{E}(B)_t\bigg(x_0 + \int_0^t \mathcal{E}(B)_s^{-1}\mu_s\mathrm{d}s\bigg),\ t \geq 0.$$

In particular, if $x_0 \geq 0$ and μ is valued in $\mathbb{R}_+$, show that X is also valued in $\mathbb{R}_+$.

2) Fix now $(x_1, x_2) \in \mathbb{R}^2$, as well as two maps a_1 and a_2 from $\mathbb{R}_+ \times \mathbb{R}$ to $\mathbb{R}$ which are Lipschitz continuous and with linear growth with respect to their second variable, uniformly in the first one. Assume that $a_1 \geq a_2$ and $x_1 \geq x_2$. Show that there are unique solutions to the SDEs

$$X_t^i = x_i + \int_0^t a_i(s, X_s^i)\mathrm{d}s + \int_0^t X_s^i\mathrm{d}B_s,\ t \geq 0,\ i \in \{1, 2\},$$

and that $X^1 \geq X^2$.

1) **Once again, we have here an SDE with uniformly Lipschitz-continuous coefficients, so existence and uniqueness of a strong solution is immediate. Then it suffices to apply Itô's formula recalling that**

$$\mathrm{d}\mathcal{E}(B)_t = \mathcal{E}(B_t)\mathrm{d}B_t,$$

to verify that the solution is given as in the statement. The non-negativity is then obvious.

$2)$ **This questions is about a technique called linearisation. First, the assumptions made ensure that X^1 and X^2 are well-defined as unique strong solutions to their respective SDEs. Then, the point is to notice that we can always write**

$$a_1(s,X_s^1)-a_2(s,X_s^2)=a_1(s,X_s^1)-a_2(s,X_s^1)+\lambda_s(X_s^1-X_s^2),$$

where

$$\lambda_s:=\frac{a_2(s,X_s^1)-a_2(s,X_s^2)}{X_s^1-X_s^2}1_{\{X_s^1\neq X_s^2\}},$$

is a bounded, measurable and $\mathbb{F}$-adapted process by the Lipschitz property of a_2. Writing $\delta X:=X^1-X^2$, we get

$$\delta X_t=(x_1-x_2)+\int_0^t\big(a_1(s,X_s^1)-a_2(s,X_s^1)+\lambda_s\delta X_s\big)\mathrm{d}s+\int_0^t\delta X_s\mathrm{d}B_s.$$

If we now write $Y_t:=\mathrm{e}^{-\int_0^t\lambda_s\mathrm{d}s}\delta X_t$, we get

$$Y_t=(x_1-x_2)+\int_0^t\mathrm{e}^{-\int_0^s\lambda_u\mathrm{d}u}\big(a_1(s,X_s^1)-a_2(s,X_s^1)\big)\mathrm{d}s+\int_0^t Y_s\mathrm{d}B_s,$$

and it then suffices to apply $1)$ with $\mu_s:=\mathrm{e}^{-\int_0^s\lambda_u\mathrm{d}u}\big(a_1(s,X_s^1)-a_2(s,X_s^1)\big)$ which is non-negative by assumption.